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PEAK SETS OF U-ALGEBRAS

Jonathan Kane

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Abstract: Let B_n be the open unit ball in $\mathbb{C}^n$ with boundary S_{2n-1}. For $n \geq 2$ and $\mu \in [0,1]$, E_μ is an algebra of functions on S_{2n-1} which is invariant under composition with unitary transformations. E_o is the well known example of continuous functions on S_{2n-1} which have continuous extensions to the closure of B_n that are holomorphic on B_n. This paper presents several results about peak sets for the algebras E_μ for $\mu > 0$. The results generalize some of the results known about peak sets for E_o.

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Section 1 - Introduction

For $n \geq 1$ let B_n be the open unit ball in $\mathbb{C}^n$ and S_{2n-1} be its boundary. For $n \geq 2$ and $\mu \in (0,1]$ let E_μ^* be the smallest subalgebra of $C(S_{2n-1})$ which contains the constant functions and the restrictions to S_{2n-1} of each monomial $z^a\bar{z}^b$ where a and b are n-dimensional multi-indices satisfying $|b| < \mu\, |a|$. Let $E_o^* = \bigcap_{\mu>0} E_\mu^*$ be the algebra of holomorphic polynomials restricted to S_{2n-1}. For $\mu \in [0,1]$ let E_μ be the closure in $C(S_{2n-1})$ of E_μ^*. Nagel and Rudin show in [8] that each E_μ is a U-algebra, that is, E_μ is closed under composition with the n-dimensional unitary transformations, $U(n)$. In particular, E_o is the algebra of functions on S_{2n-1} which have continuous extensions to $cl(B_n)$ that are holomorphic in B_n. E_1 is the algebra of functions on S_{2n-1} which have continuous extensions to $cl(B_n)$ that are holomorphic on each disk $\{\gamma z \mid \gamma \in B_1\}$ where $z \in S_{2n-1}$.

If E is a commutative Banach algebra with maximal ideal space K, then a set $P \subseteq K$ is called a peak set for E if there is an $f \in E$ such that $\hat{f}(w) = 1$ for all $w \in P$ and $|\hat{f}(w)| < 1$ for all $w \in K \setminus P$. A set $P \subseteq K$ is called an interpolation set for E if for each continuous function g on P there is an $f \in E$ with $\hat{f}(w) = g(w)$ for all $w \in P$. A problem of current research interest is that of characterizing the peak and interpolation sets for E_o. (It has been shown that for E_o, a closed set P is a peak set if and only if it is an interpolation set.) Several partial results have been obtained. In particular, see [2], [7], [9], and [10]. In this paper some of these results are extended to the algebras E_μ for $\mu > 0$. The extended results depend on the parameter μ and extend continuously to the case $\mu = 0$.

Section 2 presents some preliminary results about the maximal ideal spaces for the E_μ algebras and states three theorems concerning peak sets. The proofs of the three theorems are found in Sections 3, 4, and 5.

Section 2 - Maximal Ideal Spaces

The maximal ideal space of E_o is $cl(B_n)$, a classical result. For $f \in E_o$ and $w \in cl(B_n)$ $\hat{f}(w)$ is simply point evaluation at w of the holomorphic extension of f to $cl(B_n)$. The maximal ideal space of E_1 is also $cl(B_n)$ and for $f \in E_1$ $\hat{f}$ is the extension of f to $cl(B_n)$ that is holomorphic on each disk $\{\gamma z \mid \gamma \in B_1\}$ where $z \in S_{2n-1}$. This is a result of Hoffman and Singer [5].

The maximal ideal space of E_μ for μ strictly between 0 and 1 is more complicated and is found in [6]. A brief description of the maximal ideal space for E_μ follows. Let $A_n = \{w \in \mathbb{C}^{2n} \mid \sum_{j=1}^{n} w_j w_{n+j} = 1\}$ and $\mu \in (0,1)$. Then each $w \in A_n$ induces a multiplicative linear functional on E_μ^* by mapping each monomial $Z^a\bar{Z}^b$ to w^c where c is the 2n-dimensional multi-index with $c_j = a_j$ and $c_{n+j} = b_j$ for $j \leq n$. Essentually the functional induced by w maps each polynomial $p(z_1,z_2,\dots z_n,\bar{z}_1,\bar{z}_2,\dots \bar{z}_n) \in E_\mu^*$ to $p(w_1,w_2,\dots w_{2n})$. $w \in A_n$ insures that if $p_1 \equiv p_2$ on S_{2n-1}, then $p_1(w) = p_2(w)$. The only multiplicative linear functional of E_μ^* which does not arrise in this manner is the trivial functional which maps all non-constant monomials to 0.

The maximal ideal space of E_μ consists of all the multiplicative linear functionals of E_μ^* which are bounded on E_μ^* and thus extend to E_μ. For $r \in (0,1]$ define $\sigma(r,\mu) = \dfrac{\mu\lambda^{-1} + \lambda^{-\mu}}{1 + \mu}$ where λ is defined implicitly by $r = \dfrac{\mu\lambda + \lambda^{\mu}}{1 + \mu}$. For $w \in \mathbb{C}^{2n}$ let

$r(w) = \sum_{j=1}^{n} |w_j|^2$ and $s(w) = \sum_{j=1}^{n} |w_{n+j}|^2$. Define

$K_\mu = \{w \in A_n \mid r(w) \in (0,1) \text{ and } s(w) < \sigma(r(w),\mu)\}$. The maximal ideal space of E_μ is the one point compactification of $cl(K_\mu)$. For $f \in E_\mu^*$ and $w \in cl(K_\mu)$ $\hat{f}(w)$ is evaluated as discussed in the last paragraph. If w is the point at infinity in the one point compactification of $cl(K_\mu)$, then the functional induced by w is the trivial functional.

Let $S = \{w \in cl(K_\mu) \mid r(w) = 1\}$. For $w \in S$, $w_j = \bar{w}_{n+j}$ for $j \leq n$. Let $\pi : \mathbb{C}^{2n} \to \mathbb{C}^n$ be the projection onto the first n coordinates. The following proposition contains several facts about K_μ found in [6].

PROPOSITION 1

If $\mu \in (0,1)$, then

a) for $\mu \leq \mu' < 1$ $K_{\mu'} \subseteq K_\mu$.

b) $\pi(K_\mu) = B_n \setminus \{0\}$.

c) for $z \in B_n \setminus \{0\}$ the set $\{w \in K_\mu \mid \pi(w) = z\}$ is an n-dimensional ball intersected with A_n.

d) if $\{w_1, w_2, w_3 \dots\} \subseteq K_\mu$ with $r(w_k) \to 0$, then $s(w_k) \to \infty$.

e) π is a bijection from S onto S_{2n-1}.

f) $cl(K_\mu)$ is polynomially convex.

g) for $f \in E_\mu$, $\hat{f}$ is holomorphic on K_μ and $\hat{f}(w) = f(\pi(w))$ for each $w \in S$.

h) S is the Shilov boundary of the maximal ideal space of E_μ.

i) E_μ is isomorphic to the algebra of all functions continuous on the one point compactification of $cl(K_\mu)$ and holomorphic on K_μ.

Results found in [2], [7], [9], and [10] show that a smooth curve or manifold P lying in S_{2n-1} is a peak set for E_o if and only if it also satisfies certain direction conditions. Theorems 1, 2, and 3 of this paper extend some of these results to the algebras E_μ for $\mu > 0$. The direction conditions in these theorems depend continuously on the parameter μ.

For the algebra E_μ with $\mu \in (0,1)$ the problem is to find which sets $P \subseteq S$ are peak or interpolation sets. If G is the inverse of the projection $\pi : S \to S_{2n-1}$, the problem can be restated as finding those sets $P \subseteq S_{2n-1}$ such that $G(P)$ is a peak or interpolation set. One approach to this problem is to find all real analytic submanifolds, $P \subseteq S_{2n-1}$, which have the property that every real analytic function on $G(P)$ is the restriction of a function f holomorphic in a neighborhood of $cl(K_\mu)$. Such a manifold is called an analytic interpolation manifold for E_μ.

If P is a submanifold of S_{2n-1} and $z \in P$, let $T_z(P)$ be the tangent space of P at z. In [2] Burns and Stout show that a real analytic manifold $P \subseteq S_{2n-1}$ is an analytic interpolation manifold for E_o if and only if for all $z \in P$ and $w \in T_z(P)$, $\langle w,z\rangle = 0$. (Their result applies more generally to the algebra of functions continuous on $cl(D)$ and holomorphic on D where D is any smooth strictly pseudoconvex domain. Their result does not apply to E_μ since K_μ is not smooth at S.) The methods of Burns and Stout can be applied to obtain the following result.

THEOREM 1 Let P be a closed, totally real, real analytic submanifold of S_{2n-1} and let $\mu \in (0,1)$.

a) If for all $z \in P$ and $w \in T_z(P)$ with $w \neq 0$, $|\langle w,z\rangle| < \frac{2\sqrt{\mu}}{1+\mu}|w|$, then P is an analytic interpolation manifold for E_μ.

b) If for some $z \in P$ and $w \in T_z(P)$ $|\langle w,z\rangle| > \frac{2\sqrt{\mu}}{1+\mu}|w|$, then P is not an analytic interpolation manifold for E_μ. Moreover, $G(P)$ is not a peak or interpolation set for E_μ.

The real analytic smoothness condition of Theorem 1 can be greatly reduced. This has been done for E_o by Nagel, Rudin, and Wainger in [9] and [10]. These ideas will be applied to the algebras E_μ.

If γ is a map from $\mathbb{R}^1$ to $\mathbb{R}^n$ which is m times differentiable and there exist positive constants $\alpha < 1$ and A such that $\left|\frac{d^m\gamma(x_1)}{dx^m} - \frac{d^m\gamma(x_2)}{dx^m}\right| \leq A|x_1 - x_2|^\alpha$ for all $x_1, x_2 \in \mathbb{R}^1$, then γ is said to satisfy a <u>Lipschitz condition of order $\alpha + m$</u> or that γ belongs to the class $\mathrm{Lip}(\alpha + m)$.

<u>THEOREM 2</u> Let $\gamma : [0,1] \to S_{2n-1}$ satisfy a Lipschitz condition of order $\alpha > 2$. Suppose that for all $x \in (0,1)$, $|\mathrm{Im}\langle\gamma'(x),\gamma(x)\rangle| > \frac{2\sqrt{\mu}}{1+\mu}|\gamma'(x)|$. Then if D is a closed subset of $[0,1]$ with positive Lebesque measure, $G(\gamma(D))$ is not a peak set for E_μ.

As μ increases to 1, the condition $|\mathrm{Im}\langle\gamma'(x),\gamma(x)\rangle| > \frac{2\sqrt{\mu}}{1+\mu}|\gamma'(x)|$ becomes $|\mathrm{Im}\langle\gamma'(x),\gamma(x)\rangle| = |\gamma'(x)|$ or $\gamma'(x) = k\gamma(x)i$. This suggests that a curve γ might be a peak set for E_1 if $\gamma(x)$ and $\gamma'(x)$ do not point in the same direction.

<u>THEOREM 3</u> A closed set $P \subseteq S_{2n-1}$ is a peak set for E_1 if and only if for each $x \in S_{2n-1}$ the set of points, ζ, in S_1 with $\zeta x \in P$ has Lebesque measure zero (Lebesque measure on S_1.)

Section 3 - Proof of Theorem 1

Proofs of Theorems 1 and 2 will depend on the following properties of the function $\sigma(r,\mu)$ which can be provided by direct calculation.

<u>PROPOSITION 2</u>

For $r \in (0,1]$ and $\mu \in (0,1)$

a) $\sigma(r,\mu)$ is real analytic and decreasing in r and μ.

b) $\sigma(1,\mu) = 1$.

c) $\frac{\partial\sigma}{\partial r} < -1$ for $r \in (0,1)$.

d) $\frac{\partial^2 \sigma}{\partial r^2} > 0$ for $r \in (0,1)$.

e) $\sigma(r,\mu) = 1 - (r - 1) + \frac{(1 + \mu)^2}{4\mu}(r - 1)^2 - \frac{(1 + \mu)^4}{16\mu^2}(r - 1)^3 + O((r - 1)^4)$.

The proof of Theorem 1 will use the notion of the complexification of a real analytic manifold. If $M \subseteq \mathbb{C}^n$ is a totally real, real analytic manifold, then there is an associated complex manifold $M^* \subseteq \mathbb{C}^n$ containing M called the complexification of M. If M has real dimension m and γ is a coordinate map from a neighborhood of 0 in $\mathbb{R}^m$ onto a neighborhood of $\gamma(0)$ in M, then γ can be extended to be holomorphic and one-to-one in some neighborhood of 0 in $\mathbb{C}^m$. Then $\gamma(z) \in M$ if and only if z is real. If this extension is done for each coordinate patch of M, the result is the complex manifold M^*. M^* is not uniquely determined but the germ of M^* on M is unique. It is the fact that M is totally real that allows M^* to be embedded in $\mathbb{C}^n$ rather than some higher dimensional space. Note that any real analytic function on M can be extended to be holomorphic in a neighborhood of M in M^*. For more theory about complexifications see [1], [11], and [13].

If P is a closed, totally real, real analytic submanifold of S_{2n-1}, then $G(P)$ is a closed, totally real, real analytic submanifold of S with complexification $G(P)^* \subseteq \mathbb{C}^{2n}$. Following the argument used by Burns and Stout in [2] if the condition of part (a) of Theorem 1 holds, it will be shown that there is a neighborhood of $G(P)$ in $G(P)^*$ which intersects $cl(K_\mu)$ only on $G(P)$. Then any real analytic function defined on $G(P)$ can be extended to a neighborhood of $G(P)$ in $G(P)^*$ and that function can be extended to a neighborhood of $cl(K_\mu)$. On the other hand if the condition of part (b) of Theorem 1 is satisfied, it will be shown that every neighborhood of $G(P)$ in $G(P)^*$ intersects K_μ and, therefore, there is a real analytic function on $G(P)$ which cannot be extended to be holomorphic on K_μ.

For each $z \in P$ there is a regular one-to-one map, γ, from a neighborhood of 0 in $\mathbb{R}^m$ onto a neighborhood of z in P with

$\gamma(0) = z$ and $\gamma_j(w) = \sum \frac{\partial^a \gamma_j(0)}{\partial w^a} \frac{w^a}{a!}$. Let ψ map a neighborhood of 0 in $\mathbb{R}^m$ to $G(P)$ in $\mathbb{C}^{2n}$ by $\psi_j(w) = \gamma_j(w)$ and $\psi_{n+j}(w) = \overline{\gamma_j(w)}$ for $j \le n$. Then ψ can be extended to a neighborhood of 0 in $\mathbb{C}^m$ so that ψ is a biholomorphic map onto a neighborhood of $G(z)$ in $G(P)^*$. To see if $G(P)^*$ intersects K_μ, one can calculate $r(\psi(w))$, $s(\psi(w))$, and $\sigma(r(\psi(w)),\mu)$.

$$r(\psi(w)) = \sum_{j=1}^{n} |\psi_j(w)|^2 = \sum_{j=1}^{n} |\gamma_j(w)|^2 = \langle\gamma(w),\gamma(w)\rangle =$$

$$\langle\sum \frac{\partial^a\gamma(0)}{\partial w^a}\frac{w^a}{a!}, \sum \frac{\partial^a\gamma(0)}{\partial w^a}\frac{w^a}{a!}\rangle = \langle\gamma(0),\gamma(0)\rangle +$$

$$2\ \mathrm{Re}\langle \sum_{k=1}^{m} \frac{\partial\gamma(0)}{\partial w_k} w_k, \gamma(0)\rangle + \sum_{k,\ell=1}^{m} \langle\frac{\partial\gamma(0)}{\partial w_k} w_k, \frac{\partial\gamma(0)}{\partial w_\ell} w_\ell\rangle +$$

$$\mathrm{Re} \sum_{k,\ell=1}^{m} \langle\frac{\partial^2\gamma(0)}{\partial w_k \partial w_\ell} w_k w_\ell, \gamma(0)\rangle + O(|w|^3)$$

$$= 1 + 2\ \mathrm{Re} \sum_{k=1}^{m} w_k\langle\frac{\partial\gamma(0)}{\partial w_k}, \gamma(0)\rangle + \sum_{k,\ell=1}^{m} w_k \overline{w_\ell}\langle\frac{\partial\gamma(0)}{\partial w_k}, \frac{\partial\gamma(0)}{\partial w_\ell}\rangle +$$

$$\mathrm{Re} \sum_{k,\ell=1}^{m} w_k w_\ell\langle\frac{\partial^2\gamma(0)}{\partial w_k \partial w_\ell}, \gamma(0)\rangle + O(|w|^3).$$

Similarly,

$$s(\psi(w)) = \sum_{j=1}^{n} |\psi_{n+j}(w)|^2 = \langle\sum \overline{\frac{\partial^a\gamma(0)}{\partial w^a}}\frac{w^a}{a!}, \sum \overline{\frac{\partial^a\gamma(0)}{\partial w^a}}\frac{w^a}{a!}\rangle =$$

$$1 + 2\ \mathrm{Re} \sum_{k=1}^{m} w_k\overline{\langle\frac{\partial\gamma(0)}{\partial w_k}, \gamma(0)\rangle} + \sum_{k,\ell=1}^{m} w_k \overline{w_\ell}\ \overline{\langle\frac{\partial\gamma(0)}{\partial w_k}, \frac{\partial\gamma(0)}{\partial w_\ell}\rangle} +$$

$$\mathrm{Re} \sum_{k,\ell=1}^{m} w_k w_\ell\overline{\langle\frac{\partial^2\gamma(0)}{\partial w_k \partial w_\ell}, \gamma(0)\rangle} + O(|w|^3).$$

From Proposition 2e $\sigma(r(\psi(w)),\mu) =$

$$1 - 2\,\mathrm{Re} \sum_{k=1}^{m} w_k\left\langle\frac{\partial\gamma(0)}{\partial w_k},\gamma(0)\right\rangle - \sum_{k,\ell=1}^{m} w_k\bar{w}_\ell\left\langle\frac{\partial\gamma(0)}{\partial w_k}, \frac{\partial\gamma(0)}{\partial w_\ell}\right\rangle -$$

$$\mathrm{Re} \sum_{k,\ell=1}^{m} w_k w_\ell\left\langle\frac{\partial^2\gamma(0)}{\partial w_k\partial w_\ell},\gamma(0)\right\rangle$$

$$+ \frac{(1+\mu)^2}{4\mu}\left[2\,\mathrm{Re} \sum_{k=1}^{m} w_k\left\langle\frac{\partial\gamma(0)}{\partial w_k},\gamma(0)\right\rangle\right]^2 + O(|w|^3).$$

Using the fact that $\mathrm{Re}\left\langle\frac{\partial\gamma(0)}{\partial w_k}, \gamma(0)\right\rangle = 0$ and, therefore,

$\mathrm{Re}\left\langle\frac{\partial^2\gamma(0)}{\partial w_k\partial w_k},\gamma(0)\right\rangle = -\mathrm{Re}\left\langle\frac{\partial\gamma(0)}{\partial w_k}, \frac{\partial\gamma(0)}{\partial w_\ell}\right\rangle$ yields

$$r(\psi(w)) = 1 - 2\,\mathrm{Im}\left\langle \sum_{k=1}^{m} \mathrm{Im}(w_k)\frac{\partial\gamma(0)}{\partial w_k},\gamma(0)\right\rangle + O(|w|^2) \quad \text{and}$$

$$s(\psi(w)) - \sigma(r(\psi(w)),\mu) = 2 \sum_{k,\ell=1}^{m} [\mathrm{Re}(w_k\bar{w}_\ell) - \mathrm{Re}(w_k w_\ell)]\,\mathrm{Re}\left\langle\frac{\partial\gamma(0)}{\partial w_k}, \frac{\partial\gamma(0)}{\partial w_\ell}\right\rangle -$$

$$\frac{(1+\mu)^2}{4\mu}\left[-2\,\mathrm{Im}\left\langle \sum_{k=1}^{m} \mathrm{Im}(w_k)\frac{\partial\gamma(0)}{\partial w_k},\gamma(0)\right\rangle\right]^2 + O(|w|^3).$$

If for small w $\mathrm{Im}\left\langle \sum_{k=1}^{m} \mathrm{Im}(w_k)\frac{\partial\gamma(0)}{\partial w_k}, \gamma(0)\right\rangle < 0$, then $r(\psi(w)) > 1$ and $\psi(w)$ is not in $cl(K_\mu)$.

If the condition of part (a) is satisfied, then there is an $\eta > 0$ such that for all $z \in P$ and $w \in T_z(P)$ with $w \neq 0$,

$|\langle w,z\rangle| < \sqrt{1-\eta}\,\frac{2\sqrt{\mu}}{1+\mu}|w|$. Therefore, when $\mathrm{Im}\left\langle \sum_{k=1}^{m} \mathrm{Im}(w_k)\frac{\partial\gamma(0)}{\partial w_k},\gamma(0)\right\rangle \geq 0$ for small w, $r(\psi(w)) \leq 1$ but $s(\psi(w)) - \sigma(r(\psi(w)),\mu)$

$$> 2 \sum_{k,\ell=1}^{m} [\mathrm{Re}(w_k\overline{w}_\ell) - \mathrm{Re}(w_k w_\ell)] \,\mathrm{Re}\left\langle \frac{\partial\gamma(0)}{\partial w_k}, \frac{\partial\gamma(0)}{\partial w_\ell}\right\rangle -$$

$$\frac{(1+\mu)^2}{4\mu} 4(1-\eta) \frac{4\mu}{(1+\mu)^2} \left| \sum_{k=1}^{n} \mathrm{Im}(w_k) \frac{\partial\gamma(0)}{\partial w_k}\right|^2$$ which simplifies to

$4\eta \left| \sum_{k=1}^{m} \mathrm{Im}(w_k) \frac{\partial\gamma(0)}{\partial w_k}\right|^2$. Since $\{\frac{\partial\gamma(0)}{\partial w_k} \mid k \leq m\}$ is linearly independent when P is total real, $s(\psi(w)) > \sigma(r(\psi(w)),\mu)$ if w small and $w \in \mathbb{C}^n \backslash \mathbb{R}^m$. Thus $w \in \mathbb{C}^m \backslash \mathbb{R}^m$ implies that $\psi(w) \notin \mathrm{cl}(K_\mu)$ and so there is a neighborhood of $G(P)$ in $G(P)^*$ which intersects $\mathrm{cl}(K_\mu)$ only on $G(P)$. Now continuing with the argument of Burns and Stout if g is a real analytic function defined on $G(P)$, g can be extended to g^*, a holomorphic function on a neighborhood of $G(P)$ in $G(P)^*$. Since $\mathrm{cl}(K_\mu)$ is polynomally convex, there is an open polynomial polyhedron, H, containing $\mathrm{cl}(K_\mu)$ such that g^* is holomorphic on the analytic subvariety $G(P)^* \cap H$ which is closed in H. Since H is a domain of holomorphy, there is a function f holomorphic in H such that $f(z) = g^*(z)$ for $z \in G(P)^* \cap H$. This shows that P is an analytic interpolation manifold.

On the other hand if the condition of part (b) is satisfied, then there is a $z_0 \in P$, $w \in T_{z_0}(P)$, and $\eta > 0$ such that

$|\langle w,z_0\rangle| > \sqrt{1+\eta}\,\frac{2\sqrt{\mu}}{1+\mu}|w|$. In particular, there is a suitably small w so that $\mathrm{Im}\langle \sum_{k=1}^{m} \mathrm{Im}(w_k) \frac{\partial\gamma(0)}{\partial w_k},\gamma(0)\rangle > 0$ so $r(\psi(w)) < 1$ and $s(\psi(w)) - \sigma(r(\psi(w)),\mu) < -4\eta \left| \sum_{k=1}^{m} \mathrm{Im}(w_k) \frac{\partial\gamma(0)}{\partial w_k}\right|^2$. Thus

$s(\psi(w)) < \sigma(r(\psi(w)),\mu)$ so $\psi(w) \in K_\mu$. Suppose $\mathrm{Im}(w_j) \neq 0$. Let θ be the j-th coordinate function of the inverse of ψ defined in a neighborhood of z_0 in $G(P)$. Then $g(z) = 1/[\theta(z) - \theta(\psi(w))]$ is real analytic in a neighborhood of z_0 in $G(P)$ so g can be extended to be real analytic on $G(P)$. In fact g can be extended to be real analytic on all of $\mathbb{C}^{2n}$ be a result of Cartan [3; page 89]. But now since g cannot be extended from $G(P)$ to be holomorphic on $G(P)^*$, there can be no f holomorphic in a neighborhood of $cl(K_\mu)$ which extends g. This shows that P is not an analytic interpolation manifold for E_μ.

Moreover, any function continuous on $cl(K_\mu)$ and holomorphic on K_μ which is identically 1 on $G(P)$, must be 1 on all of $G(P)^* \cap K_\mu \neq \phi$. Thus $G(P)$ is not a peak set. $G(P)$ is not an interpolation set since the function g cannot be extended. This completes the proof of Theorem 3.

Section 4 - Proof of Theorem 2

Theorem 2 will follow from the following two intermediate results.

PROPOSITION 3 Let μ and μ' be given with $0 < \mu < \mu' < 1$. There is a positive constant A (depending on μ and μ') so that whenever $z \in K_{\mu'}$, the distance from z to the boundary of K_μ is greater than $A[1 - r(z)]^2$.

THEOREM 4 Let $\gamma : [0,1] \to S_{2n-1}$ satisfy a Lipschitz condition of order $\alpha > 2$. Suppose that there exists $\eta > 0$ such that

$$|\operatorname{Im}\langle\gamma'(x),\gamma(x)\rangle| \geq (1 + \eta)\frac{2\sqrt{\mu}}{1+\mu}|\gamma'(x)| \quad \text{for } x \in (0,1).$$

Then there is a continuous map $\Gamma : [0,1] \times [0,1] \to cl(K_\mu)$ with $\Gamma(x,0) = G(\gamma(x))$ for all $x \in [0,1]$ and $\Gamma(x,y) \in K_\mu$ for all $y > 0$ such that for each bounded holomorphic function, h, on K_μ, $\lim_{y\to 0} h(\Gamma(x,y))$ exists for almost every $x \in [0,1]$.

Proof of Proposition 3

It will be convenient to represent the points $w \in \mathbb{C}^{2n}$ by the $2 \times n$ matrix $J_w = \begin{bmatrix} w_1 & w_2 & \cdots & w_n \\ \hline w_{n+1} & w_{n+2} & \cdots & w_{2n} \end{bmatrix}$. For unitary matrix $v \in U(n)$

define $T_V : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ by $T_V(z) = w$ if $J_w = J_z V$. Then T_V is biholomorphic with $T_V^{-1} = T_{V^{-1}}$. Clearly, for $w \in \mathbb{C}^{2n}$ $\pi(T_V(w)) = \pi(w)V$, $r(T_V(w)) = r(w)$, and $s(T_V(w)) = s(w)$. T_V is an automorphism of $cl(K_\mu)$, that is, T_V is a continuous bijection of $cl(K_\mu)$ onto itself which is holomorphic on K_μ. In fact all automorphisms of $cl(K_\mu)$ arrise as T_V for some $V \in U(n)$ although this result will not be proved here. Note that for each $w \in cl(K_\mu)$, there are $u \in (0,1]$, $v \geq 0$, and $V \in U(n)$ so that

$$J_w V = \begin{bmatrix} u & 0 & 0 & \cdots & 0 \\ u^{-1} & v & 0 & \cdots & 0 \end{bmatrix},$$

$r(w) = u^2$, and $s(w) = u^{-2} + v^2$.

Let μ and μ' be given and let z be a fixed point in the boundary of $K_{\mu'}$. Let D be the distance from z to the boundary of K_μ which means $D^2 = \inf \sum_{j=1}^{n} |z_j - w_j|^2$ where w ranges over the boundary of K_μ. Since the problem is invariant under translations T_V for $V \in U(n)$,
$D^2 = \inf\{ | u,0,\ldots 0) - (u',0,\ldots 0)V |^2$
$+ | (u^{-1},v,0,\ldots 0) - (u'^{-1},v',0,\ldots 0)V |^2\}$ where $u^2 = r(z)$,
$u^{-2} + v^2 = s(z)$, $V \in U(n)$, and $J_w = \begin{bmatrix} u' & 0 & 0 & \cdots & 0 \\ u'^{-1} & v' & 0 & \cdots & 0 \end{bmatrix}$ for some w in the boundary of K_μ. From this
$D^2 \geq \inf\{ | (u,0,\ldots 0) - (u',0,\ldots 0)V_1 |^2 +$
$| (u^{-1},v,0,\ldots 0) - (u'^{-1},v',0,\ldots 0)V_2 |^2\} = \inf\{ | u - u' |^2 +$
$| \sqrt{u^{-2} + v^2} - \sqrt{u'^{-2} + v'^2} |^2\} = \inf | \{\sqrt{r(z)} - \sqrt{r(w)} |^2 + | \sqrt{s(z)} - \sqrt{s(w)} |^2\}$
where w ranges over the boundary of K_μ. It is sufficient to prove the propositon for z close to S, so it can be assumed that there is a constant $c > 0$ so that $\sqrt{r(z)} + \sqrt{r(w)} \leq c$ and $\sqrt{s(z)} + \sqrt{s(w)} \leq c$. Thus $D^2 \geq c^{-1} \inf\{[r(z) - r(w)]^2$
$+ [s(z) - s(w)]^2\} = c^{-1} \inf\{[r(z) - x]^2 + [s(z) - y]^2\}$ where $(r(z), s(z))$ is a fixed point on the graph of $y = \sigma(x,\mu')$ and (x,y) is a variable point on the graph of $y = \sigma(x,\mu)$. Since $r(z)$ and x are close to 1, Proposition 2 shows that there are constants a and b with $\frac{(1 + \mu)^2}{4\mu} > a > b > \frac{(1 + \mu')^2}{4\mu'}$ so that

$\sigma(x,\mu) > 2 - x + a(x-1)^2 > 2 - x + b(x-1)^2 > \sigma(x,\mu')$ for all x in question. Thus $D^2 \geq c^{-1} \inf_{x \leq 1} \{[r(z) - x]^2 + [r(z) - x + a(x-1)^2 - b(r(z)-1)^2]^2\}$. If $|r(z) - x| < \frac{1}{2}(a-b)(r(z)-1)^2$, then
$D^2 \geq c^{-1}[r(z) - x + a(x-1)^2 - b(r(z)-1)^2]^2 =$
$c^{-1}\{(r(z)-x)[1 + a(2 - r(z) - x)] + (r(z)-1)^2(a-b)\}^2 \geq$
$c^{-1}[(a-b)(r(z)-1)^2 - |r(z)-x|]^2 \geq \frac{(a-b)^2}{4c}(r(z)-1)^4$.
Otherwise if $|r(z) - x| \geq \frac{1}{2}(a-b)(r(z)-1)^2$, then
$D^2 \geq c^{-1}[r(z)-x]^2 \geq \frac{(a-b)^2}{4c}(r(z)-1)^4$ and so in either case there is an $A > 0$ so that $D \geq A(r(z)-1)^2$ which proves the proposition.

Proof of Theorem 4

The argument is essentually one used by Nagel and Rudin in [9] adapted to $cl(K_\mu)$. $\gamma \in Lip(\alpha)$ for $\alpha > 2$ so $\gamma'(x)$ and $\gamma''(x)$ are continuous and $\gamma''(x) \in Lip(\alpha - 2)$. Extend $\gamma''(x)$ so that it is a function still in $Lip(\alpha - 2)$ but is now a continuous function of compact support in $\mathbb{R}^1$. Let $P_y(x) = \frac{1}{\pi}\frac{y}{x^2 + y^2}$ and $u(x,y) = \int_{\mathbb{R}^1} \gamma''(x-t)\, P_y(t)\, dt$. For $j \leq n$ define

$\psi_j(x,y) = \gamma_j(x) + iy\gamma_j'(x) - \frac{1}{2}\, y^2 u_j(x,y)$ and

$\psi_{n+j}(x,y) = \overline{\gamma_j(x)} + iy\overline{\gamma_j'(x)} - \frac{1}{2}\, y^2\overline{u_j(x,y)}$. Then for $j \leq n$, $\psi_j(x,y) = \gamma_j(x) + O(y)$ and $\psi_{n+j} = \overline{\gamma_j(x)} + O(y)$ as $y \to 0$.
$\frac{\partial\psi_j}{\partial x}(x,y) = \gamma_j'(x) + iy\gamma_j''(x) - \frac{1}{2}\, y^2\,\frac{\partial u_j(x,y)}{\partial x} = \gamma_j'(x) + O(y)$ since $\left|\frac{\partial u_j(x,y)}{\partial x}\right| \leq Ay^{\alpha-3}$ for some constant A. This follows from [12; pp. 141-145]. Similary, $\frac{\partial\psi_{n+j}}{\partial x}(x,y) = \overline{\gamma_j'(0)} + O(y)$.
$\frac{\partial\psi_j}{\partial y}(x,y) = i\gamma_j'(x) - \frac{1}{2}\, y^2\,\frac{\partial u_j}{\partial y}(x,y) - yu_j(x,y) = i\gamma_j'(x) + O(y)$ and

$\frac{\partial\psi_{n+j}}{\partial y}(x,y) = i\overline{\gamma_j'(x)} + O(y)$. Then $\frac{\partial\psi_j}{\partial \bar z}(x,y) = \frac{\partial\psi_j}{\partial x}(x,y)$ $+ i\frac{\partial\psi_j}{\partial y}(x,y) = iy[\gamma''(x) - u_j(x,y)] - \frac{1}{2}y^2\frac{\partial u_j}{\partial \bar z}(x,y)$.

$|\gamma_j''(x) - u_j(x,y)| \le Ay^{\alpha-2}$ and $|\frac{\partial u_j}{\partial \bar z}(x,y)| \le Ay^{\alpha-3}$ for some A so $|\frac{\partial\psi_j}{\partial \bar z}(x,y)| = O(y^{\alpha-1})$. Similarly $|\frac{\partial\psi_{n+j}}{\partial \bar z}(x,y)| = O(y^{\alpha-1})$.

Continuing, $\frac{\partial^2\gamma_j}{\partial y^2}(x,y) = -y\frac{\partial u_j}{\partial y}(x,y) - \frac{1}{2}y^2\frac{\partial^2 u_j}{\partial y^2}(x,y) - u_j(x,y) =$ $-\gamma_j''(x) + O(y^{\alpha-2})$ and $\frac{\partial^2\psi_{j+n}}{\partial y^2} = -\overline{\gamma_j''(x)} + O(y^{\alpha-2})$. So taking limits as $y \to 0$ yields $\psi_j(x,0) = \gamma_j(x), \psi_{n+j}(x,0) = \overline{\gamma_j(x)}$, $\frac{\partial\psi_j}{\partial y}(x,0) = i\gamma_j'(x)$, $\frac{\partial\psi_{n+j}}{\partial y}(x,0) = i\overline{\gamma_j'(x)}$, $\frac{\partial^2\psi_j}{\partial y^2}(x,0) = -\gamma_j''(x)$, and $\frac{\partial^2\psi_{n+j}}{\partial y^2}(x,0) = -\overline{\gamma_j''(x)}$.

Since $\sum_{j=1}^{n} \psi_j(x,0)\,\psi_{n+j}(x,0) = 1$ for all $x \in [0,1]$, there is an $\varepsilon_1 > 0$ such that $|\sum_{j=1}^{n} \psi_j(x,y)\,\psi_{n+j}(x,y)| > 1/2$ for all $y \in [0,\varepsilon_1]$ and $x \in [0,1]$. For $(x,y) \in [0,1] \times [0,\varepsilon_1]$ define

$$\Gamma_j(x,y) = \psi_j(x,y) \text{ and } \Gamma_{n+j}(x,y) = \frac{\psi_{n+j}(x,y)}{\sum_{k=1}^{n} \psi_k(x,y)\psi_{n+k}(x,y)} \text{ for } j \le n.$$

Note that $\sum_{j=1}^{n} \Gamma_j(x,y)\Gamma_{n+j}(x,y) \equiv 1$ for $(x,y) \in [0,1] \times [0,\varepsilon_1]$.

Now it is claimed that there is an $\varepsilon_2 \in (0,\varepsilon_1)$ such that for some $\mu' > \mu$, $\Gamma(x,y) \in K_{\mu'}$ for all $(x,y) \in [0,1] \times (0,\varepsilon_2)$. To prove this, calculate $r(\Gamma(x,y))$, $s(\Gamma(x,y))$, and $\sigma(r(\Gamma(x,y)),\mu)$.

$$r(\Gamma(x,y)) = \sum_{j=1}^{n} \Gamma_j\overline{\Gamma_j}(x,y) \cdot \sum_{j=1}^{n} \Gamma_j\overline{\Gamma_j}(x,0) = 1,$$

$$\frac{\partial}{\partial y}\sum_{j=1}^{n}\Gamma_j\overline{\Gamma_j}(x,0) = 2\,\mathrm{Re}(i\sum_{j=1}^{n}\gamma_j'(x)\overline{\gamma_j(x)}) = -2\,\mathrm{Im}\langle\gamma'(x),\gamma(x)\rangle,$$

and $\frac{\partial^2}{\partial y^2}\sum_{j=1}^{n}\Gamma_j\overline{\Gamma_j}(x,0) = \sum_{j=1}^{n}[-2\,\mathrm{Re}(\gamma_j''(x)\overline{\gamma_j(x)}) + 2|\gamma_j'|^2) =$ $2(|\gamma'(x)|^2 - \mathrm{Re}\langle\gamma''(x),\gamma(x)\rangle) = 4|\gamma'(x)|^2$ so $r(\Gamma(x,y)) =$ $1 - 2\,\mathrm{Im}\langle\gamma'(x),\gamma(x)\rangle y + 2|\gamma'(x)|^2 y^2 + o(y^2)$. By hypothesis, $\mathrm{Im}\langle\gamma'(x),\gamma(x)\rangle$ is never 0 so it can be assumed that $\mathrm{Im}\langle\gamma'(x),\gamma(x)\rangle > 0$. Then for small $y > 0$, $r(\Gamma(x,y)) < 1$. Now,

$$s(\Gamma(x,y)) = \sum_{j=1}^{n}\Gamma_{n+j}\overline{\Gamma_{n+j}}(x,y) = \frac{\sum_{j=1}^{n}\psi_{n+j}\overline{\psi_{n+j}}(x,y)}{(\sum_{j=1}^{n}\psi_j\psi_{n+j}(x,y))(\sum_{j=1}^{n}\overline{\psi_j\psi_{n+j}(x,y)})} =$$

$\frac{p(y)}{q(y)}$. $\sum_{j=1}^{n}\Gamma_{n+j}\overline{\Gamma_{n+j}}(x,0) = 1$, $\frac{\partial}{\partial y}\sum_{j=1}^{n}\Gamma_{n+j}\overline{\Gamma_{n+j}}(x,0) =$

$\frac{q(0)p'(0) - p(0)q'(0)}{[q(0)]^2} = p'(0)$ since $q'(0) = 0$ and $p(0) = q(0) = 1$.

$$\frac{\partial}{\partial y}\sum_{j=1}^{n}\Gamma_{n+j}\overline{\Gamma_{n+j}}(x,0) = 2\,\mathrm{Re}(i\sum_{j=1}^{n}\overline{\gamma_j'(x)}\gamma_j(x)) = 2\,\mathrm{Im}\langle\gamma'(x),\gamma(x)\rangle .$$

$$\frac{\partial^2}{\partial y^2}\sum_{j=1}^{n}\Gamma_{n+j}\overline{\Gamma_{n+j}(x,0)} =$$

$\{q^2(0)[q(0)p''(0) - p(0)q''(0)] - q'(0)2q(0)[q(0)p'(0) - p(0)q'(0)]\}/$ $[q(0)]^4 = p''(0) - q''(0) = 2\sum_{j=1}^{n}[-\mathrm{Re}(\overline{\gamma_j''(x)}\gamma_j(x) +$ $|\gamma_j'(x)|^2] = 2[|\gamma'(x)|^2 - \mathrm{Re}\langle\gamma''(x),\gamma(x)\rangle] = 4|\gamma'(x)|^2$. So $s(\Gamma(x,y)) = 1 + 2\,\mathrm{Im}\langle\gamma'(x),\gamma(x)\rangle\, y + 2|\gamma'(x)|^2 y^2 + o(y^2)$. Finally,

$$\sigma(r(\Gamma(x,y)),\mu) = 1 - (r-1) + \frac{(1+\mu)^2}{4\mu}(r-1)^2 + O((r-1)^3) =$$

$1 + 2\ \mathrm{Im}\langle\gamma'(x),\gamma(x)\rangle\ y - 2\,|\gamma'(x)|^2\ y^2 + \frac{(1+\mu)^2}{4\mu}\ 4(\mathrm{Im}\langle\gamma'(x),\gamma(x)\rangle)^2\ y^2$ $+ o(y^2)$. Thus $\sigma(r(\Gamma(x,y)),\mu) - s(\Gamma(x,y)) = -4\,|\gamma'(x)|^2\ y^2 + \frac{(1+\mu)^2}{4\mu}\ 4(\mathrm{Im}\langle\gamma'(x),\gamma(x)\rangle)^2 + o(y^2)$. But by hypothesis this is greater than or equal to $-4\,|\gamma'(x)|^2\ y^2 + 4(1+\eta)\,|\gamma'(x)|^2 +$ $o(y^2) = 4\eta\,|\gamma'(x)|^2\ y^2 + o(y^2)$. Thus for all small $y > 0$ $\Gamma(x,y) \in K_\mu$. Moreover, there is a $\mu' > \mu$ such that $\Gamma(x,y) \in K_{\mu'}$ for small y.

It now follows from Propositon 2 that there is a constant $A > 0$ so that for $(x,y) \in [0,1] \times (0,\varepsilon_2)$, the distance from $\Gamma(x,y)$ to the boundary of K_μ is greater than $A\ y^2$. It is now claimed that every function, h, which is bounded and holomorphic on K_μ possesses limits $\lim_{y\to 0} h(\Gamma(x,y))$ for almost every $x \in [0,1]$.

Suppose h is holomorphic on K_μ and that for some $M > 0$,
$|h(z)| \leq M$ for all $z \in K_\mu$. Then $|\nabla h(\Gamma(x,y))| \leq \frac{M}{A\ y^2}$. Now

$$\left|\frac{\partial}{\partial \bar z}(h(\gamma(x,y)))\right| = \left|\langle\nabla h(\Gamma(x,y)), \frac{\partial}{\partial \bar z}\Gamma(x,y)\rangle\right| \leq \frac{M}{A\ y^2}\cdot O(y^{\alpha-1}) =$$

$O(y^{\alpha-3})$. Since $\alpha - 3 > -1$, $\left|\frac{\partial}{\partial \bar z}(h(\Gamma(x,y))\right| \in L^p([0,1] \times [0,\varepsilon_2])$ for some $p > 1$. Consider the following lemma proven by Nagel and Rudin.

<u>LEMMA 1</u> Let f be a bounded continuously differentiable function on $Q = (0,1) \times (0,\varepsilon)$ for some $\varepsilon > 0$ such that $\frac{\partial f}{\partial \bar z} \in L^p(Q)$. Then $\lim_{y\to 0} f(x + iy)$ exists for almost every $x \in (0,1)$.

This lemma proves the claim. So for each h, bounded and holomorphic on K_μ, $\lim_{y\to 0} h(\Gamma(x,y))$ exists for almost every $x \in (0,1)$. Theorem 4 now follows by a change of variables.

Theorem 2 now follows from Theorem 4 exactly as done in [9]. Without loss of generality assume that for some $\eta > 0$

$|\operatorname{Im}\langle\gamma'(x),\gamma(x)\rangle| > (1+\eta)\frac{2\sqrt{\mu}}{1+\mu}|\gamma'(x)|$ for $x \in (0,1)$ for if this is false just redefine γ on a subinterval of $[0,1]$ where the inequality does hold. Suppose that there is a function $f \in E_\mu$ such that $|\hat{f}(z)| < 1$ for $z \in cl(K_\mu)\backslash G(\gamma(D))$ and $\hat{f}(z) = 1$ for $z \in G(\gamma(D))$. Then on K_μ $\operatorname{Re}(1-\hat{f}) > 0$ so there exists a function $g = \log(1-\hat{f})$ holomorphic on K_μ. Note that $\operatorname{Im} g$ is bounded and $\operatorname{Re} g(z) \to -\infty$ if $z \to G(\gamma(D))$. Let $h = e^{ig}$. Since $\operatorname{Re}(ig)$ is bounded, h is a bounded holomorphic function on K_μ. If $z \to G(\gamma(D))$, $\operatorname{Im}(ig(z)) \to -\infty$ so $h(z)$ cannot have a limit. But this h contradicts the conclusion of Theorem 4. Therefore, no such function f exists and $G(\gamma(D))$ is not a peak set of E_μ.

Section 5 - Proof of Theorem 3

Let π be the continuous projection from $\mathbb{C}^n\backslash\{0\}$ onto $\mathbb{C}P(n-1)$ where $\pi(x) = \pi(y)$ if and only if $x = \alpha y$ for some $\alpha \in \mathbb{C}^1$. For $x \in S_{2n-1}$ and $\Omega \subset \mathbb{C}^n$ define $c_x(\Omega) = \{\zeta \in \mathbb{C}^1 \mid \zeta x \in \Omega\}$. Let λ be Lebesque measure on S_1. It must be shown that a closed set $P \subseteq S_{2n-1}$ is a peak set for E_1 if and only if for each $x \in S_{2n-1}$ $\lambda(c_x(P)) = 0$. The necessity of $\lambda(c_x(P)) = 0$ is clear for if $f(\zeta x)$ were holomorphic for $\zeta \in B_1$ and $f(\zeta x) = 1$ for $\zeta \in c_x(P)$, then $\lambda(c_x(P)) > 0$ would imply that $f(\zeta x) = 1$ for all $\zeta \in cl(B_1)$.

The proof of sufficiency will construct a function in E_1 which peaks on the set P. The following results found in [4] will be used.

<u>LEMMA 2</u> Let w map B_1 conformally onto K where K is bounded by a rectifiable Jordan curve. Then

(1) $w(\zeta)$ is continuous on $cl(B_1)$ and absolutely continuous on S_1.

(2) $w'(\zeta) \in H^1$.

(3) $\dfrac{dw(e^{i\theta})}{d\theta} = ie^{i\theta}w'(e^{i\theta})$ almost everwhere on S_1.

(4) the length of arc $s(\theta',\theta'')$ on $z = w(e^{i\theta})$ between $w(e^{i\theta'})$ and $w(e^{i\theta''})$ is given by $s(\theta',\theta'') = \int_{\theta'}^{\theta''} |w'(e^{i\theta})|\,d\theta$.

LEMMA 3 Let $\{K_m\}$ be a sequence of connected domains in $\mathbb{C}^1$ with $0 \in K_m$ such that each K_m is bounded by a Jordan curve γ_m. Suppose K_m converges to a domain K bounded by a Jordan curve γ with $0 \in K_m$. Let f_m map B_1 conformally onto K_m with $f_m(0) = 0$ and $f_m'(0) > 0$. Then f_m converges uniformly on $cl(B_1)$ to a function f with $f(0) = 0$, $f'(0) > 0$, and f maps B_1 conformally onto K if and only if for each $\varepsilon > 0$ there is an $M > 0$ such that for all $m \geq M$ there is a correspondence between points of γ_m and γ such that the distances between corresponding points of γ and γ_m are less than ε.

Let P be a closed subset of S_{2n-1} such that $\lambda(c_x(P)) = 0$ for each $x \in S_{2n-1}$. Let $q(x)$ be a real valued function on S_{2n-1} which is continuously differentiable and so $q(x) = 1$ for $x \in P$ and $q(x) > 1$ for $x \notin P$. Let $\Omega = \{z \in \mathbb{C}^n \mid z = \alpha x$ for some $x \in S_{2n-1}$ and $\alpha \in \mathbb{R}$ with $0 \leq \alpha < q(x)\}$. Then Ω is an open set containing $cl(B_n)\backslash P$ and containing P in its boundary.

For $x \in S_{2n-1}$ define $\Omega_x = c_x(\Omega)$. Let f_x be the conformal map of B_1 onto Ω_x with $f_x(0) = 0$ and $f_x'(0) > 0$. By Lemma 2 f_x extends to a map of $cl(B_1)$ onto $cl(\Omega_x)$ and by Lemma 3 $f_x(\zeta)$ is continuous in x as well as in ζ.

For $y \in \mathbb{C}P(n - 1)$ select x with $\pi(x) = y$ and define the measure μ_y on $\mathbb{C}^n$ by $\mu_y(D) = \lambda(\{\zeta \in S_1 \mid f_x(\zeta)x \in D\})$. μ_y is well-defined for all $y \in \mathbb{C}P(n - 1)$ because λ is rotationally invariant. It follows from Lemma 2 that μ_y is absolutely continuous with respect to λ and thus $\mu_y(P) = 0$ for each $y \in \mathbb{C}P(n - 1)$.

Define G_m to be the open set in $\mathbb{C}^n$ given by $\{x \in \mathbb{C}^n \mid dist(x,P) < 1/m\}$ for m a postive inteter. Since μ_y is a regular measure for each $y \in \mathbb{C}P(n - 1)$, $\lim_{m\to\infty} \mu_y(G_m) = \mu_y(P) = 0$.

For each $\varepsilon > 0$ and $y \in \mathbb{C}P(n - 1)$ define $k_\varepsilon(y) = \min\{m \mid \mu_y(G_m) < \varepsilon\}$.

LEMMA 4 For each $\varepsilon > 0$, $k_\varepsilon(y)$ is a bounded function on $\mathbb{C}P(n - 1)$.

Proof of Lemma 4

Let $y_0 \in \mathbb{C}P(n-1)$ with $k_\varepsilon(y_0) = M$ and let W_1 be a neighborhood of y_0 in $\mathbb{C}P(n-1)$ and W_2 be a set in S_{2n-1} such that $\pi : W_2 \to W_1$ is a homeomorphism of W_2 onto W_1. Let $h : W_1 \to W_2$ be the inverse to the homeomorphism. $f_x(\zeta)$ is continuous in x and ζ so $f_{h(y)}(\zeta) : cl(B_1) \times W_1 \to \mathbb{C}^1$ is continuous and $f_{h(y)}(\zeta) \cdot h(y) : cl(B_1) \times W_1 \to cl(\Omega)$ is continuous. Choose $\delta > 0$ so that $|y - y_0| < \delta$ implies that

$$dist(f_{h(y)}(\zeta) \cdot h(y), f_{h(y_0)}(\zeta) \cdot h(y_0)) < \frac{1}{2M} \text{ for all } \zeta \in cl(B_1).$$

Then for all y such that $|y - y_0| < \delta$, $f_{h(y)}(\zeta)h(y) \in G_{2M}$ for some $\zeta \in S_1$ implies that $dist(f_{h(y_0)}(\zeta)h(y_0),P) \leq dist(f_{h(y_0)}(\zeta)h(y_0), f_{h(y)}(\zeta)h(y)) + dist(f_{h(y)}(\zeta)h(y),P) \leq \frac{1}{2M} + \frac{1}{2M} = \frac{1}{M}$ so $f_{h(y_0)}(\zeta)h(y_0) \in G_M$. Thus

$$\{\zeta \in S_1 \mid dist(f_{h(y)}(\zeta) \cdot h(y),P) < \tfrac{1}{2M}\} \subseteq \{\zeta \in S_1 \mid dist(f_{h(y_0)}(\zeta) \cdot h(y_0),P) < \tfrac{1}{M}\}.$$

So $|y - y_0| < \delta$ implies $\mu_y(G_{2M}) \leq \mu_{y_0}(G_M) < \varepsilon$ so $k_\varepsilon(y) \leq 2M$. Thus $k_\varepsilon(y)$ is locally bounded which implies $k_\varepsilon(y)$ is bounded since $\mathbb{C}P(n-1)$ is compact. This proves the lemma.

Select m_j so that $\mu_y(G_{m_j}) < \frac{1}{2^j}$ for all $y \in \mathbb{C}P(n-1)$ and so $m_{j+1} > m_j$. Let $a_j : \mathbb{C}^n \to [0,1]$ be a continuous function such that $a_j(x) = 1$ for $x \in G_{m_{j+1}}$ and $a_j(x) = 0$ for $x \in \mathbb{C}^n \setminus G_{m_j}$. Let $b_m(x) = \sum_{j=1}^{m} a_j(x)$ and $b(x) = \sum_{j=1}^{\infty} a_j(x)$. Then $b_m(x)$ converges uniformly to $b(x)$ on $cl(\Omega)\setminus G_m$ for any fixed M. Thus

$$\mu_y(\{x \mid j - 1 \leq b(x) \leq j\}) \leq \mu_y(G_{m_j} \setminus G_{m_{j-1}}) \leq \mu_y(G_{m_j}) < \frac{1}{2^j}.$$

$b_m \to b$ monotonically and $b(x) \to \infty$ as $x \to P$.

LEMMA 5

(1) For all $x \in S_{2n-1}$, $b(f_x(\zeta) \cdot x) \in L^p(S_1)$ for $p \geq 1$.

(2) For $p \geq 1$ $b(f_x(\zeta) \cdot x) : S_{2n-1} \to L^p(S_1)$ is a continuous function of x.

Proof of Lemma 5

(1) follows from $\int_{S_1} |b(f_x(\zeta)x)|^p \, d\lambda(\zeta) =$

$$\sum_{j=1}^{\infty} \int_{j-1<|b(f_x(\zeta)\cdot x)|\le j} |b(f_x(\zeta)\cdot x)|^p \, d\lambda(\zeta) \le \sum_{j=1}^{\infty} \frac{j^p}{2^j} < \infty.$$

To prove (2) let $\varepsilon \in (0,1)$ be given. Choose M such that $2^p \sum_{j=M}^{\infty} \frac{j^p}{2^j} < \frac{\varepsilon}{2}$. Choose $\delta_1 > 0$ so that $|x - x'| < \delta_1$ implies

$\mathrm{dist}(f_x(\zeta)x, f_{x'}(\zeta)x') < (\frac{1}{m_{M+1}} - \frac{1}{m_{M+2}})$ for all $\zeta \in S_1$. Then if $f_x(\zeta)x \notin G_{m_{M+1}}$, $f_{x'}(\zeta)x' \notin G_{m_{M+2}}$. b is uniformly

continuous on $\mathrm{cl}(\Omega)\backslash G_{M+2}$ so choose $\delta_2 \in (0,\delta_1)$ such that

$|b(f_x(\zeta)\cdot x) - b(f_{x'}(\zeta)x')|^p < \frac{\varepsilon}{4\pi}$ for all $\zeta \in S_1$ with $f_x(\zeta)x \in G_{m_{M+1}}$ and for $|x - x'| < \delta_2$. Then $|x - x'| < \delta_2$ implies

$$\int_{S_1} |b(f_x(\zeta)x) - b(f_{x'}(\zeta)x')|^p \, d\lambda(\zeta) \le \int_{f_x(\zeta)x \notin G_{m_{M+1}}} |b(f_x(\zeta)x) - b(f_{x'}(\zeta)x')|^p \, d\lambda(\zeta) + \int_{\substack{f_x(\zeta)x \in G_{m_M} \text{ and} \\ f_{x'}(\zeta)x' \in G_{m_M}}} |b(f_x(\zeta)x) - b(f_{x'}(\zeta)x')|^p \, d\lambda(\zeta) \le \frac{1}{2\pi}\int_{S_1} \frac{\varepsilon}{2} \, d\lambda(\zeta) +$$

$$\int_{\substack{f_x(\zeta)x \in G_{m_M} \text{ and} \\ f_{x'}(\zeta)x' \in G_{m_M}}} 2^{p-1}[|b(f_x(\zeta)x)|^p + |b(f_{x'}(\zeta)x')|^p] \, d\lambda(\zeta) \le \varepsilon/2 + 2p \sum_{j=M}^{\infty} \frac{j^p}{2^j} < \varepsilon.$$

This proves (2).

For $x \in S_{2n-1}$, $b(f_x(\zeta)x) \in L^1(S_1)$ so there is a function $g_x(\zeta)$ with $g_x(\zeta) = b(f_x(\zeta)x)$ for $\zeta \in S_1$, g_x harmonic in B_1, and

$g_x(\zeta) = \lim_{r\to 1} g_x(r\zeta)$ for almost every $\zeta \in S_1$. Then g_x is continuous at all points of $cl(B_1)$ where g_x is finite. If $x \in S_{2n-1}$ and $\zeta \in cl(B_1)$, define $G(x\zeta) = g_x(\zeta) - g_x(0)$.

LEMMA 6 G is a well-defined function on $cl(\Omega)\setminus P$ such that G is continuous on Ω and $G(z) \to \infty$ as $z \to P$ in Ω.

Proof of Lemma 6

Suppose $z \in cl(\Omega)\setminus P$ and $z = x_1\zeta_1 = x_2\zeta_2 \neq 0$. Then for some $\alpha \in S_1$, $\alpha x_1 = x_2$. By definition of f_x, $f_{\alpha x_1}(\zeta) = \alpha^{-1}f_x(\alpha\zeta)$. Thus, for all $\zeta \in S_1$, $g_{x_1}(\zeta) = b(f_{x_1}(\zeta)x_1) = b(\alpha f_{\alpha x_1}(\alpha^{-1}\zeta)x_1) = b(f_{x_2}(\alpha^{-1}\zeta)x_2) = g_{x_2}(\alpha^{-1}\zeta)$. Thus, this holds for all $\zeta \in cl(B_1)$ so if $\alpha x_1 = x_2$, $\zeta_1 = \alpha\zeta_2$ and $g_{x_1}(\zeta_1) = g_{x_2}(\zeta_2)$. Since $g_{x_1}(0) = g_{x_2}(0)$ is clear, G is well-defined.

$b(f_x(\zeta)x) : S_{2n-1} \to H^2(B_1)$ is continuous so for all $\zeta \in B_1$ so $g_x(\zeta)$ is continuous in x. Therefore, $g_x(\zeta)$ is continuous on $S_{2n-1} \times B_1$ and G is, therefore, continuous on Ω. Finally, let $P_r(\theta) = \dfrac{1 - r^2}{1 - 2r\cos\theta + r^2}$ be the Poisson kernel for B_1. Then by the monotone convergence theorem $G(\zeta x) = G(re^{i\theta}x) =$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} b(f_x(e^{i\phi})x)\, P_r(\theta - \phi)\, d\phi =$$

$$\lim_{m\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} b_m(f_x(e^{i\phi})x)\, P_r(\theta - \phi)\, d\phi.$$

$\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} b_m(f_x(e^{i\phi})x)\, P_r(\theta - \phi)\, d\phi$ is continuous on $S_{2n-1} \times cl(B_1)$ and is increasing in m so its limit is lower semi-continuous. Therefore, $G(z)$ is lower semi-continuous on $cl(\Omega)$ and since $G(z) = \infty$ for $z \in P$, $P \subseteq G^{-1}((m,\infty))$ which is open. So as $z \to P$, $G(z) \to \infty$. This proves the lemma.

$g_x(\zeta) - g_x(0)$ is harmonic in B_1 and has it boundary values in $L^2(S_1)$ which depend continuously on x. Thus there is a unique function h_x such that $[g_x(\zeta) - g_x(0)] + ih_x(\zeta)$ is holomorphic in B_1 and $h_x(0) = 0$. Since $g_x(\zeta) - g_x(0)$ is continuous as a function from S_{2n-1} to $L^2(S_1)$, $h_x(\zeta)$ depends continuously on x for each $\zeta \in B_1$. This follows from the fact that the Hilbert transform is a continuous map from H^2 to H^2. Define for $z \in \Omega$, $H(z) = h_x(\zeta)$ whenever $z = \zeta x$ for $\zeta \in B_1$ and $x \in S_{2n-1}$. $H(z)$ is then well-defined for the same reasons $G(z)$ was and $H(z)$ is continuous on Ω. Let $K(z) = G(z) + iH(z)$. Then K is continuous on Ω and for each $x \in S_{2n-1}$ $K(\zeta x)$ is holomorphic in ζ for $\zeta \in B_1$. Also $\mathrm{Re}(K(z)) \to \infty$ as $z \to P$. Since $G(z)$ is lower semi-continuous, there is a number $N > 0$ so $G(z) > -N$ for $z \in cl(B_n)\backslash P$. Note that K is continuous on $cl(B_n)\backslash P$. Let $f(z) = \frac{K(z)}{N + K(z)}$. Then $f(z)$ is continuous on $cl(B_n)$, $f(\zeta x)$ is holomorphic in ζ for each $x \in S_{2n-1}$, $f(z) = 1$ for $z \in P$, and $|f(z)| < 1$ for $z \in cl(B_n)\backslash P$. This proves the theorem.

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Department of Mathematics and Computer Science,
University of Wisconsin-Whitewater,
Whitewater, WI 53190

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